Modeling stresses on satellites due to nonsynchronous rotation and orbital eccentricity using gravitational potential theory

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The tidal stress at the surface of a satellite is derived from the gravitational potential of the satellite’s parent planet, assuming that the satellite is fully differentiated into a silicate core, a global subsurface ocean, and a decoupled, viscoelastic lithospheric shell. We consider two types of time variability for the tidal force acting on the shell: one caused by the satellite’s eccentric orbit within the planet’s gravitational field (diurnal tides), and one due to nonsynchronous rotation (NSR) of the shell relative to the satellite’s core, which is presumed to be tidally locked. In calculating surface stresses, this method allows the Love numbers h and ℓ, describing the satellite’s tidal response, to be specified independently; it allows the use of frequency-dependent viscoelastic rheologies (e.g. a Maxwell solid); and its mathematical form is amenable to the inclusion of stresses due to individual tides. The lithosphere can respond to NSR forcing either viscously or elastically depending on the value of the parameter \(\Delta \equiv \Delta / \eta\), where \(\mu\) and \(\eta\) are the shear modulus and viscosity of the shell respectively, and \(\omega\) is the NSR forcing frequency. \(\Delta\) is proportional to the ratio of the forcing period to the viscous relaxation time. When \(\Delta \gg 1\) the response is nearly fluid; when \(\Delta \ll 1\) it is nearly elastic. In the elastic case, tensile stresses due to NSR on Europa can be as large as \(-3\) MPa, which dominate the \(-50\) kPa stresses predicted to result from Europa’s diurnal tides. The faster the viscous relaxation the smaller the NSR stresses, such that diurnal stresses dominate when \(\Delta \gtrsim 100\). Given the uncertainty in current estimates of the NSR period and of the viscosity of Europa’s ice shell, it is unclear which tide should be dominant. For Europa, tidal stresses are relatively insensitive both to the rheological structure beneath the ice layer and to the thickness of the icy shell. The phase shift between the tidal potential and the resulting stresses increases with \(\Delta\). This shift can displace the NSR stresses longitudinally by as much as 45° in the direction opposite of the satellite’s rotation.

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1. Introduction

A body subject to a varying gravitational potential will experience tidal deformation as different portions of the body are subjected to different gravitational forcing. The stresses arising from tidal deformations will be either stored elastically, relieved through material failure, or relaxed away viscously. Both failure and relaxation are dissipative processes capable of doing significant work; elastic storage of stress is reversible. In a viscoelastic body, the partitioning of stress among elastic storage, failure, and relaxation will depend on the strength and rheological properties of the body, and on the period of the forcing potential. If the forcing period is roughly equal to or greater than the natural viscous relaxation time of the material being forced, significant viscous relaxation could result, preventing or reducing the extent of material failure and reducing the amount of stress stored elastically.

In the case of natural satellites there are many possible sources of time-variable tidal deformation, for example tidal despinning (Melosh, 1977, 1980b), reorientation relative to the spin axis (Melosh, 1975, 1980a), orbital recession or procession (Squyres and Croft, 1986; Helfenstein and Parmentier, 1983), nonsynchronous rotation (Helfenstein and Parmentier, 1985), polar wander (Leith and McKinnon, 1996), and radial and libra-
tional tides due to an eccentric orbit (Yoder, 1979; Hoppa, 1998; Greenberg et al., 1998). Many satellites exhibit large-scale systems of linear surface features that have been interpreted as faults, fractures, and other tectonic structures, and which have been linked to the above mechanisms. Examples include the three resonant Galilean satellites (McEwen et al., 2004; Greeley et al., 2004; Pappalardo et al., 2004), several of the middle-sized uranian and saturnian satellites (Squares and Croft, 1986; Croft and Soderblom, 1991; Nimmo and Pappalardo, 2006; Nimmo et al., 2007), Triton (Croft et al., 1995), and even Mars' small irregular satellite Phobos (Dobrovolskis, 1982). In these cases, observations of a satellite's global tectonic features, combined with a model of its tidal deformation, can be used to gain understanding of the satellite's dynamical and structural evolution.

In the case of icy satellites, viscou effects are likely to play an important role in the stress environment. This is because viscosity generally drops significantly as a material approaches its melting point, and the melting point of ice is much lower than that of silicates. Additionally, tidal heating appears to be an important factor in the histories of many icy bodies (Ojakangas and Stevenson, 1989; Meyer and Wisdom, 2007; Showman et al., 1997), meaning portions of the icy moons may have spent significant periods of time at relatively high temperatures.

In this work, we develop a model of tidal deformation and stress for an arbitrary satellite that is based on long-standing methods of computing global tides and stresses on the Earth. We treat the lithosphere as a layered Maxwell viscoelastic solid to understand how the relaxation of tidal stresses could affect the interpretation of global tectonic features. We focus on non-synchronous rotation (NSR) and eccentricity (diurnal) tides, which offer the most plausible explanation for the pattern of lineaments observed on the surface of Europa (Helfenstein and Parmentier, 1985; McEwen, 1986; Leith and McKinnon, 1996; Geissler et al., 1998; Greenberg et al., 1998; Hoppa, 1998; Hoppa et al., 1999a, 1999b; Figueredo and Greeley, 2000; Kattenhorn, 2002; Spaun et al., 2003). We assume the satellite has radially dependent material properties, and include a rocky core, a hydrostatic ocean, and a lithospheric shell of arbitrary thickness having multiple viscosity layers. The frequency-dependent response is incorporated into the model through the use of complex-valued Lamé parameters and Love numbers. For demonstration purposes we apply the model to Jupiter's satellite Europa. We assume Europa's lithosphere consists of a high-viscosity outer layer surrounding a low viscosity inner layer, which approximates an ice shell undergoing stagnant lid convection. A parallel approach has been undertaken by Harada and Kurita (2007), who also consider a tidal potential method and viscoelastic relaxation of NSR stresses, but who do not provide the details of their calculations.

Our approach of deriving stresses directly from the gravitational potential has several advantages over previously published approaches, which derive stresses based on an instantaneous change in the triaxial ellipsoid describing a satellite's shape (Melosh, 1977; Helfenstein and Parmentier, 1985; Leith and McKinnon, 1996; Hoppa, 1998): 1. It allows the Love numbers and ℓ to be specified independently, decoupling the radial and lateral tidal deformations. 2. It allows the use of realistic rheological properties for the body (e.g., treatment of the ice shell as a Maxwell solid; radially dependent structural properties, including the presence of a solid core; compressibility), and does not require that the outer shell be thin. 3. Because viscoelastic effects are included directly into the equations of motion, the results can be applied to all possible combinations of NSR rates and viscosity values.

4. Its mathematical form is amenable to the inclusion of individual potential terms (NSR or diurnal) and the future inclusion of other terms in the description of the potential (e.g., obliquity or polar wander).

In this paper we describe the NSR and diurnal tidal forcing mechanisms (Sections 2 and 3), and develop a mathematical framework for computing the resulting outer surface stresses both for elastic (Section 4) and viscoelastic (Section 5) satellites. The effects of viscoelasticity on the NSR stresses are discussed qualitatively in Section 6. We use the viscoelastic model developed in Section 5 to compute diurnal and NSR stresses for Europa (Sections 7 and 8), and compare with results computed using previously published methods (Section 9). More mathematical detail is provided in Appendices A and B.

2. Stressing mechanisms

2.1. Nonsynchronous rotation

The tidal despinning timescales for large natural satellites are short compared to the age of the Solar System, meaning that today nearly all satellites are synchronously locked, always showing their parent planet the same hemisphere (Peale, 1999). However, many large icy satellites are believed to have global oceans which decouple the motions of their floating shells from their interiors (Schubert et al., 2004). Such a decoupled shell could experience a net tidal torque, and could rotate slightly faster than synchronously (cf. Greenberg and Weidenschilling, 1984; Ojakangas and Stevenson, 1989). From the rotating shell's point of view, the apparent location of the parent planet moves slowly across the sky. The tidal bulge, which remains fixed relative to the parent planet, appears to migrate in the direction opposing the rotation. As the bulge passes over a region of the shell, that region deforms and experiences a stress. If NSR exists, it should occur with a period similar to the thermal diffusion or viscous relaxation timescales of the shell (Greenberg and Weidenschilling, 1984; Ojakangas and Stevenson, 1989), and thus viscous relaxation is likely to influence NSR stresses. Observations constrain the present period of NSR of Europa's ice shell to be \( > 10^4 \) yr (Hoppa et al., 1999c). NSR could also affect Ganymede (Collins et al., 1998; Zahnle et al., 2003; Nimmo and Pappalardo, 2004) and Io (Greenberg and Weidenschilling, 1984; Schenk et al., 2001), and could apply to other satellites with fluid or low-viscosity interiors.

2.2. Diurnal tides

Diurnal stresses for a synchronously rotating satellite have the same period as a satellite's orbit. They arise on a satellite in an eccentric orbit for two reasons. First, for an eccentric orbit the distance between the satellite and the planet changes with time, thus changing the amplitude of the planet's gravitational force on the satellite, creating the "radial tide." At periapsis the tidal bulge is larger than at apoapsis, and this daily deformation results in diurnally varying stresses.

Second, a synchronously orbiting satellite in an eccentric orbit does not always keep the same face toward the planet. A satellite at periapsis is orbiting slightly faster than its (constant) rotation rate, and at apoapsis is orbiting slightly slower. This causes the tidal potential to rock back and forth relative to fixed points in the satellite, inducing a "librational tide" (Yoder, 1979; Greenberg et al., 1998). As will be seen below, the magnitudes of these tides are proportional to the orbital eccentricity \( e \) (for small values of \( e \)). Diurnal tides have been suggested to explain the formation of the...
3. The tidal potential

Tidal deformation is computed as the first-order departure from a tide-free equilibrium state. In the equilibrium state of a synchronously rotating satellite the entire satellite is rotating together without deforming, and the satellite’s rotation rate is equal to its mean motion,  \( n \), about the parent planet. For Europa, for example, the rotation rate is \( n = 2\pi / 3.55 \) radians/terrestrial day. We assume in this paper that tidal dissipation has driven the orbital obliquity of the satellite to zero, meaning that the satellite’s rotation axis is perpendicular to its orbital plane. Recent work (Bills, 2005) suggests that Europa may have a forced obliquity of \( \sim 0.1^\circ \), which if present could result in surface stresses comparable in magnitude to those due to Europa’s orbital eccentricity (Hurford et al., 2006). Those stresses are not considered here.

If NSR occurs, the satellite’s equilibrium state consists of the rocky core and liquid ocean rotating synchronously with the orbital motion, but with the lithospheric shell rotating slightly faster than synchronously. We assume the NSR axis coincides with the axis of synchronous rotation, and so is perpendicular to the orbital plane. In this case, the angular rotation rate of the core and ocean is \( n + b \), where \( b \) is the angular rate of NSR.

We attach a coordinate system to the outer surface of the shell, with the \( \hat{z} \) axis along the shell’s rotation axis, and with the \( \hat{x} \) axis pointing toward the parent planet at periapse (assumed to occur at time \( t = 0 \)). The coordinate system rotates with the shell, and so its rotation rate is \( n + b \). Let an arbitrary point in this coordinate system have the spherical coordinates \( r, \theta, \phi \) (radius, co-latitude, and eastward longitude, respectively). The tidal acceleration at \( (r, \theta, \phi) \) caused by the parent planet is the gradient of the tidal potential, \( V_T(r, \theta, \phi) \). The tidal potential, \( V_T \), can be expanded into spherical harmonics, as described in Appendix A. The result, when NSR is much slower than the diurnal rotation, is

\[
V_T(r, \theta, \phi, t) = Z \left( \frac{r}{R_s} \right)^2 \left[ T_s + T_0 + T_1 + T_2 \right],
\]

where

\[
T_s = \frac{1}{6} (1 - 3 \cos^2 \theta),
\]

\[
T_0 = \frac{1}{2} \sin^2 \theta \cos(2\phi + 2bt),
\]

\[
T_1 = \frac{\epsilon}{2} (1 - 3 \cos^2 \theta) \cos(nt),
\]

\[
T_2 = \frac{\epsilon}{2} \sin^2 \theta [3 \cos(2\phi) \cos(nt) + 4 \sin(2\phi) \sin(nt)].
\]

Here, \( \epsilon \) is the orbital eccentricity, \( R_s \) is the radius of the satellite, \( t \) is time relative to periapse, and the constant \( Z \) is defined in Eq. (A.3).

Suppose the satellite’s orbit is circular, so that \( \epsilon = 0 \). Then the tidal potential is represented entirely by \( T_s + T_0 \). If, in addition, there is no NSR (i.e. \( b = 0 \)), then the potential is independent of time as seen from any point in the satellite. The potential would cause a permanent tidal bulge fixed to the satellite and pointing directly toward and away from the parent planet (there are outward tidal bulges on both sides of the satellite), but there would be no tidal shear stresses associated with that bulge, since all stresses would have had an infinite time to relax.

Nonsynchronous rotation, on the other hand, causes the outer shell to rotate with angular velocity \( b \) through the maximum in the tidal potential, resulting in the 2bt time dependence shown in Eq. (3). The parent planet has longitude \( \phi = -bt \) relative to our satellite-fixed coordinate system, and so Eq. (3) describes an elongated potential pointing directly toward and away from the planet. \( T_0 \) is still time-independent, and thus does not cause shear stresses. However, because of the time dependence in \( T_0 \), viscosity within the outer shell would cause a time delay between when the shell passes through the \( V_T \) maximum, and when the shell fully deforms. Thus, the tidal bulge would not point directly toward the parent planet’s center of mass, but (as we will show) would be offset from it by an amount that depends on the nonsynchronous rotation rate and the viscosity.

The \( T_1 \) and \( T_2 \) terms in Eq. (1) represent the diurnal tidal potential. Those terms are smaller than the NSR tidal potential by a factor of eccentricity (for Europa \( \epsilon = 0.0094 \)). Thus, NSR stresses tend to be much larger than the diurnal stresses, unless the NSR period is very long compared to the relaxation time (as discussed in Section 6.1). \( T_1 \) and the \( \cos(nt) \) term in \( T_2 \) cause the radial tide, and the \( \sin(nt) \) term in \( T_2 \) causes the librational tide (see Section 2.2).

4. Tidal displacements and surface stresses for an elastic satellite

We denote the tidal displacement vector at any point \((r, \theta, \phi)\) within the satellite as \( \vec{\delta}(r, \theta, \phi) \). The vector \( \vec{\delta} \) is related to the applied tidal potential \( V_T \) through the differential equation describing conservation of linear momentum (cf. Eq. (2.2) of Wahr (1981); ignoring centrifugal and Coriolis forces):

\[
\rho \vec{\delta} = -\rho \nabla \phi^E - \rho \vec{\delta} \cdot \nabla(\rho \phi) + \nabla \cdot \tau
+ \rho \nabla \phi \left[ \left( \nabla \vec{\delta} \right) I - \left( \nabla \vec{\delta} \right)^T \right] + \rho o V_T.
\]

where \( \tau \) is the stress tensor, \( \phi^E \) is the gravitational potential arising from tidal deformation of the satellite, \( \rho \) is the density, \( \phi \) is the satellite’s equilibrium (nontidal) gravitational potential, I is the identity matrix, and the superscript \( T \) denotes transpose. Poisson’s equation, relating \( \phi^E \) to the tidal displacement field, is

\[
\nabla^2 \phi^E = -4\pi G \nabla \cdot (\rho \vec{\delta}),
\]

where \( G \) is Newton’s gravitational constant. The stress–displacement relation is

\[
\tau = \lambda (\nabla \cdot \vec{\delta}) I + \mu \left[ \nabla \vec{\delta} + (\nabla \vec{\delta})^T \right],
\]

where \( \tau \) is the stress tensor, and \( \mu \) and \( \lambda \) are the Lamé parameters (\( \mu \) is the shear modulus). These differential equations (Eqs. (6)–(8)) assume the satellite is compressible and self-gravitating. The material properties, described by \( \rho \), \( \mu \), and \( \lambda \), can vary spatially. The unknown variables, \( \vec{\delta} \), \( \tau \), and \( \phi^E \), are determined by solving these differential equations subject to continuity conditions across internal boundaries and at the outer surface.

The tidal potential, \( V_T \) (Eq. (1)), is composed solely of second-degree spherical harmonics in \( (\theta, \phi) \) \( (T_s, T_{0}, T_{1}, T_{2}) \) have degree, order \( = (2,0) \), and \( T_0 \) and \( T_2 \) have degree, order \( = (2,2) \). We assume the equilibrium state of the satellite is spherically symmetric, so that all boundaries are spheres and \( \rho \), \( \mu \), and \( \lambda \) depend only on \( r \) (i.e. they are independent of \( \theta \) and \( \phi \)). In this case, spherical harmonics separate the differential equations and boundary conditions, and so \( \vec{\delta} \) is composed of those same second-order harmonics. Specifically, \( \vec{\delta} = s \hat{r} + s \hat{\theta} + s \hat{\phi} \) at the outer surface \( (r = R_s) \) can be related to \( V_T \) at the outer surface, using the two second-degree dimensionless Love numbers, \( h \) and \( \ell \) (cf. Munk and MacDonald, 1975; Lambeck, 1980), as...
\[ s_\tau (r = R_s, \theta, \phi, t) = \left( \frac{h}{g} \right) V_T \bigg|_{r=R_s}, \quad (9) \]
\[ s_\phi (r = R_s, \theta, \phi, t) = \left( \frac{\ell}{g} \right) \frac{\partial V_T}{\partial \theta} \bigg|_{r=R_s}, \quad (10) \]
\[ s_{\phi \theta} (r = R_s, \theta, \phi, t) = \left( \frac{\ell}{g \sin \theta} \right) \frac{\partial V_T}{\partial \phi} \bigg|_{r=R_s}, \quad (11) \]

where \( g \) is the gravitational acceleration at the surface of the satellite. The Love number \( h \) describes the amplitude of the radial displacement \( s_r \), and \( \ell \) describes both the southward \( s_\theta \) and eastward \( s_{\phi \theta} \) lateral displacements.

The stress tensor associated with the tidal displacement vector is given by Eq. (8). We are concerned in this paper with modeling stresses on the satellite's outer surface. Since there are no tidal surface tractions on the outer surface, \( s_r = s_{\phi \theta} = 0 \) at the outer surface. Since \( \tau \) is symmetric, both \( \tau_{\phi \theta} \) and \( \tau_{\phi \phi} \) are also 0. Thus the only non-zero stress components at the outer surface are \( \tau_{r \phi} \) and \( \tau_{\phi \theta} \).

In Appendix B we express these components in terms of the surface displacements, \( s_r, s_\theta, \) and \( s_{\phi \theta} \). Then we use Eqs. (9)–(11) to relate those results to the Love numbers \( h \) and \( \ell \) and the tidal potential at the surface, \( V_T \). Using Eq. (1) to write the tidal potential as a function of \( h \), \( \ell \), and \( t \), the resulting stresses are

\[ \tau_{r\phi} = \frac{Z}{2 g R_s} \left[ -\frac{1}{3} (\beta_1 + 3 \gamma_1 \cos(2\theta)) \right. 
+ (\beta_1 - \gamma_1 \cos(2\theta)) \cos(2\phi + 2bt) 
+ 3\epsilon (\beta_1 - \gamma_1 \cos(2\theta)) \cos(2nt) \cos(2\phi) 
- \epsilon (\beta_1 + 3 \gamma_1 \cos(2\theta)) \cos(nt) 
+ 4\epsilon (\beta_1 - \gamma_1 \cos(2\theta)) \sin(nt) \sin(2\phi) \left. \right], \quad (12) \]

\[ \tau_{\phi \theta} = \frac{Z}{2 g R_s} \left[ -\frac{1}{3} (\beta_2 + 3 \gamma_2 \cos(2\theta)) \right. 
+ (\beta_2 - \gamma_2 \cos(2\theta)) \cos(2\phi + 2bt) 
+ 3\epsilon (\beta_2 - \gamma_2 \cos(2\theta)) \cos(2nt) \cos(2\phi) 
- \epsilon (\beta_2 + 3 \gamma_2 \cos(2\theta)) \cos(nt) 
+ 4\epsilon (\beta_2 - \gamma_2 \cos(2\theta)) \sin(nt) \sin(2\phi) \left. \right], \quad (13) \]

\[ \tau_{\phi \phi} = \frac{2 \ell Z}{g R_s} \left[ -\cos \theta \sin(2\phi + 2bt) + 4 \epsilon \sin(nt) \cos \theta \cos(2\phi) 
- 3 \epsilon \cos(nt) \cos \theta \sin(2\phi) \right]. \quad (14) \]

where

\[ \beta_1 = \mu \left[ \alpha (h - 3\epsilon) + 3\ell \right], \quad (15) \]
\[ \gamma_1 = \mu \left[ \alpha (h - 3\epsilon) - \ell \right], \quad (16) \]
\[ \beta_2 = \mu \left[ \alpha (h - 3\epsilon) - 3\ell \right], \quad (17) \]
\[ \gamma_2 = \mu \left[ \alpha (h - 3\epsilon) + \ell \right], \quad (18) \]

and

\[ \alpha = 3\lambda + 2\mu. \quad (19) \]

The first terms in Eqs. (12) and (13) (i.e. those of the form \( \frac{1}{3} (\beta_1 + 3 \gamma_1 \cos(2\theta)) \)) describe time-independent stresses. Those stresses would not exist on a satellite that has relaxed to a hydrostatic state. That relaxation is one of the consequences of viscoelasticity, discussed in the following section.

### 5. Viscoelastic surface stresses

The stress results shown in Eqs. (12)–(14) are derived under the assumption that the satellite behaves elastically. Moreover, the stress–displacement relation (Eq. (8)), and the relationships between the displacements and the tidal potential (Eqs. (9)–(11)), all of which were used to derive the stress results, are valid only for an elastic rheology. A more appropriate model for the rheology of the satellite, particularly at the long periods characterizing NSR, is a Maxwell solid. The correspondence principle (Peltier, 1974) can be applied to generalize the above results to a viscoelastic satellite, as follows.

Suppose the tidal potential is written as a sum of terms, each of which has an \( e^{i\omega t} \) time dependence, where \( i \) is the imaginary unit and \( \omega \) is the frequency. For example, the NSR (\( T_0 \)) term in Eq. (1) can be written as

\[ V_{T_0} = Z \left( \frac{r}{R_s} \right)^2 \left( \frac{1}{4} \right) \sin^2 \theta \left[ e^{i2\theta \omega} e^{i2bt} + e^{-i2\theta \omega} e^{-i2bt} \right]. \quad (20) \]

so that there are two such terms, one with \( \omega = 2bt \), and the other with \( \omega = -2bt \). The correspondence principle for a Maxwell solid implies that if the tidal potential has a time dependence given by \( e^{i\omega t} \), then the differential equations of motion are still given by Eqs. (6)–(8) except with the elastic Lamé parameters \( \mu \) and \( \lambda \) replaced by

\[ \tilde{\mu}(\omega) = \mu \left( \frac{i \omega}{\ell_0 + \frac{\pi}{\eta}} \right). \quad (21) \]

\[ \tilde{\lambda}(\omega) = \lambda \left( \frac{i \omega + \frac{\pi}{\eta}}{i \omega + \frac{\pi}{\eta}} \right). \quad (22) \]

where \( \eta \) is the viscosity (Peltier, 1974). These choices for \( \tilde{\mu} \) and \( \tilde{\lambda} \) assume that viscoelastic relaxation occurs for shear stresses, but not for bulk stresses. The Maxwell relaxation time is defined as \( \tau_M = \eta / \mu \). The forcing period is \( T = 2\pi / \omega \). For NSR, \( \omega = 2b \) is twice the NR rotation rate and so \( T \) is one half the shell's rotation period. Defining the dimensionless parameter \( \Delta \) as

\[ \Delta \equiv \frac{T}{2 \pi \tau_M} = \frac{\mu}{\eta \omega}. \quad (23) \]

Equations (21) and (22) become

\[ \tilde{\mu}(\omega) = \mu \left( \frac{1}{1-i\Delta} \right). \quad (24) \]

\[ \tilde{\lambda}(\omega) = \lambda \left( \frac{1}{1-i\Delta} \right). \quad (25) \]

Note that \( \tilde{\mu} \) and \( \tilde{\lambda} \) are complex and depend on frequency. Thus, all unknown variables \( (s, r, \phi, \phi) \) in the equations of motion (Eqs. (6)–(8)) are also complex and frequency dependent, as are the Love numbers (still defined by Eqs. (9)–(11)). The complex Love numbers have the general form:

\[ \tilde{h}\omega) = h_{re}(\omega) + i h_{im}(\omega), \quad (26) \]

\[ \tilde{\epsilon}\omega) = \epsilon_{re}(\omega) + i \epsilon_{im}(\omega), \quad (27) \]

where \( h_{re}, h_{im}, \epsilon_{re}, \) and \( \epsilon_{im} \) are real functions of frequency.

The significance of viscoelasticity depends on the value of \( \Delta \), which in turn depends on the ratio of the forcing period to the Maxwell time. If the forcing period is much shorter than the Maxwell time, then \( \Delta \ll 1 \), and the imaginary terms in Eqs. (24) and (25) become negligible. In that case, \( \tilde{\mu}(\omega) \approx \mu \) and \( \tilde{\lambda}(\omega) \approx \lambda \) so that the effects of viscoelasticity are unimportant. In the other extreme, where the forcing period is much
longer than the Maxwell time, then $\Delta \gg 1$, and so $\tilde{\mu}^*(\omega) \approx 0$ and $\tilde{\lambda}^*(\omega) \approx \lambda + 2\mu/3 = \text{the elastic bulk modulus}$. Thus, at very long periods the material cannot support shear stresses, and so behaves as a fluid.

It follows from Eqs. (21) and (22) that $\tilde{\mu}(\omega) = \tilde{\mu}^*(\omega)$ and $\tilde{\lambda}(\omega) = \tilde{\lambda}^*(\omega)$, where the superscript * denotes complex conjugation. Similarly, by using Eqs. (21) and (22) in Eqs. (6)–(8), and taking the complex conjugate of the resulting differential equations, it can be seen that $\tilde{h}(\omega) = \tilde{h}^*(\omega)$, which in turn implies that $h(\omega) = h^*(\omega)$ and $\ell(\omega) = \ell^*(\omega)$, and thus, for example, the outer surface radial displacement in response to $V_{\gamma 0}$ as given in Eq. (20), has the form (using Eq. (9))

$$s_r(\tau = R_s, \theta, \phi, t) = \frac{Z}{4g} \sin^3 \theta \left[ h(\omega = 2b)e^{i(2\phi + 2bt)} + \tilde{h}(\omega = -2b)e^{-i(2\phi + 2bt)} \right]$$

$$- \frac{Z}{2g} \sin^2 \theta \left[ \Re \left[ h(\omega = 2b)e^{i(2\phi + 2bt)} \right] \right]$$

$$= \frac{Z}{2g} \sin^2 \theta \left[ h_{re}(2b) \cos(2\phi + 2bt) - h_{im}(2b) \sin(2\phi + 2bt) \right].$$

(28)

where $\Re$ denotes the real part. Note that the imaginary part of $h$ leads to a displacement component that is out-of-phase with $V_{\gamma 0}$. Also note that while the NSR frequency is $b$, the relevant forcing frequency in this context is actually $2b$, since the NSR potential goes through two complete oscillations over the course of 360° of longitude. Thus $\Delta_{\text{NSR}} = \mu/(2b)$. (The factor of 2 in the denominator does not appear in Eqs. (2) and (3) of Harada and Kurita, 2007.)

The surface stresses for a Maxwell rheology are derived from the elastic surface stress results, Eqs. (12)–(14), following the same rationale used for the derivation of Eq. (28). The time-dependent terms are written as the sum of two e$^{i\omega t}$ terms, Eqs. (24) and (25) are used to replace $\mu$ and $\lambda$ when solving the equations of motion, and Eqs. (26) and (27) are used to represent $h$ and $\ell$. Note that both $\tau_{\theta 0}$ and $\tau_{\phi 0}$ include a time-independent term (the first terms on the right-hand sides of Eqs. (12)–(13)). Since $\omega = 0$ for these infinite-period terms, $\Delta$ is infinite, and thus $\tilde{\mu} = 0$, so there is no induced stress (since then, after replacing $\mu$ with $\tilde{\mu}$, $\beta_1 = \gamma_1 = \beta_2 = \gamma_2 = 0$). The remaining terms reduce to

$$\tau_{\theta 0} = \frac{Z}{2gR_s} \left[ \left( \tilde{\beta}_1(2b) - \gamma_1(2b) \cos(2\phi) \right) e^{i(2\phi + 2bt)} + 3e \left( \tilde{\beta}_1(n) - \gamma_1(n) \cos(2\theta) \right) e^{i(2\phi + 2bt)} \right]$$

$$- 4e \left( \tilde{\beta}_1(n) - \gamma_1(n) \cos(2\theta) \right) e^{i(2\phi + 2bt)}$$

$$\tau_{\phi 0} = \frac{Z}{2gR_s} \left[ \left( \tilde{\gamma}_2(2b) - \beta_2(2b) \cos(2\phi) \right) e^{i(2\phi + 2bt)} + 3e \left( \tilde{\gamma}_2(n) - \beta_2(n) \cos(2\theta) \right) e^{i(2\phi + 2bt)} \right]$$

$$- 4e \left( \tilde{\gamma}_2(n) - \beta_2(n) \cos(2\theta) \right) e^{i(2\phi + 2bt)}$$

$$\tau_{\theta p} = \tau_{\phi p} = \frac{2Z}{gR_s} \left[ \left( \tilde{\Gamma}(2b) e^{i(2\phi + 2bt)} \cos \theta \right) - 3e \left( \tilde{\Gamma}(n) e^{i(2\phi + 2bt)} \cos \theta \right) \right]$$

$$- 4e \left( \tilde{\Gamma}(n) e^{i(2\phi + 2bt)} \cos \theta \right)$$

(29–31)

where

$$\beta_1 = \tilde{\mu} \left[ \bar{h}(\tilde{h} - 3\tilde{\ell}) + 3\tilde{\ell} \right] = \mu \left[ \frac{\alpha(\tilde{h} - 3\tilde{\ell})}{1 - i\alpha \Delta/3} + \frac{3\tilde{\ell}}{1 - i\Delta} \right].$$

$$\gamma_1 = \tilde{\mu} \left[ \bar{h}(\tilde{h} - 3\tilde{\ell}) - \tilde{\ell} \right] = \mu \left[ \frac{\alpha(\tilde{h} - 3\tilde{\ell})}{1 - i\alpha \Delta/3} - \frac{\tilde{\ell}}{1 - i\Delta} \right].$$

(32–33)

To derive the right-hand sides of Eqs. (32)–(35), we used

$$\beta_2 = \tilde{\mu} \left[ \bar{h}(\tilde{h} - 3\tilde{\ell}) - 3\tilde{\ell} \right] = \mu \left[ \frac{\alpha(\tilde{h} - 3\tilde{\ell})}{1 - i\alpha \Delta/3} - \frac{3\tilde{\ell}}{1 - i\Delta} \right].$$

$$\gamma_2 = \tilde{\mu} \left[ \bar{h}(\tilde{h} - 3\tilde{\ell}) + \tilde{\ell} \right] = \mu \left[ \frac{\alpha(\tilde{h} - 3\tilde{\ell})}{1 - i\alpha \Delta/3} + \frac{\tilde{\ell}}{1 - i\Delta} \right].$$

(34–35)

$$\tilde{\Gamma} = \tilde{\mu} \tilde{\ell} = \mu \tilde{\ell} = \frac{\mu \tilde{\ell}}{1 - i\Delta}.$$
 modeling tidal stresses on viscoelastic icy satellites

6.2. NSR tides in the elastic limit \( \Delta \ll 1 \)

Next, suppose the NSR period is much shorter than the relaxation timescale of the icy shell, so that \( \Delta \ll 1 \). In that case \( \mu \approx \mu \) and the shell behaves approximately elastically. The shell responds instantaneously to tidal forces, and so the orientation of the tidal bulge is still symmetric about the line to the planet. But now the shell supports shear stresses which try to hold back the bulging fluid ocean, and so the shell’s displacement is smaller than in the \( \Delta \gg 1 \) case. However, as long as the shell is thin (i.e., its thickness is much smaller than the satellite’s radius) it has only a minimal effect on the tidal bulge (see Moore and Schubert, 2000; Wu et al., 2001; Wahr et al., 2006; Rappaport et al., 2008), and so the displacements are not affected much by the strength of the shell (i.e., the Love numbers \( \tilde{h} \) and \( \tilde{\ell} \) are relatively insensitive to the value of \( \Delta \)). Fixed points in the shell still rise and fall as they non-symmetrically rotate through the bulge, and since \( \mu \) is no longer small this periodic motion now results in significant shear stress within the shell. That shear stress is trying (but failing) to confine the ocean. The shear stress is symmetrically distributed about the satellite–planet vector, since the shell is responding instantaneously. The rocky core is still oriented toward the parent planet; it does not participate in the NSR, and so the forcing remains constant at every point in the core. Thus the core still responds to the tidal force as though it were a fluid.

6.3. NSR tides in a truly viscoelastic case \( \Delta \approx 1 \)

Finally, consider an intermediate case where the NSR period and the viscous relaxation times are approximately equal (i.e. \( \Delta \approx 1 \)). The icy shell still holds back the ocean slightly since the shell can partially support shear stresses. But because the shell is viscous it does not respond instantaneously to the tidal force. By the time a portion of the shell experiences its maximum displacement, it has rotated slightly beyond the satellite–planet vector. Thus, the bulge in the icy shell is slightly ahead of the satellite–planet vector. The impact of these shear stresses on the shell’s displacement is still small, assuming the thickness of the shell is small relative to the radius of the satellite.

Figs. 1a and 1b show how the real and imaginary parts of the Love numbers vary with \( \Delta \). Notice that the real parts of the Love numbers are about the same for all values of \( \Delta \), and the imaginary parts are always small. Thus the shell’s tidal bulge is still closely aligned with the satellite–planet vector. The shell, in effect, still mostly just rides up and down on the underlying ocean during its nonsynchronous rotation. Its tidal displacement field is determined almost entirely by the shape of the ocean surface, and has little dependence on the shell’s rheological properties.

The shell’s rheology does have a significant impact on the shear stresses caused by that displacement field. Viscoelasticity in the shell can cause the shear stresses to be offset from the satellite–planet vector by up to \( \sim 45^\circ \). For example, when \( \Delta = 1, \tilde{\mu} \) and \( \tilde{\alpha} \) in Eqs. (32)–(36) have real and imaginary parts that are of the same order, and so they cause \( \beta_1 \) and the other complex coefficients to have significant imaginary parts even though (for a thin shell) the Love numbers \( \tilde{h} \) and \( \tilde{\ell} \) are almost real. (Figs. 1c and 1d show how the real and imaginary parts of \( \tilde{\mu} \) and \( \lambda \) vary with \( \Delta \).)

A perhaps counterintuitive result is that the stress pattern ends up being shifted in the direction opposite of the shell’s rotation, rather than in the same direction. This can be seen by noting that the real and imaginary parts of Eqs. (32)–(36) have the same sign, for \( \tilde{h} \) and \( \tilde{\ell} \) real. The stress pattern shifts in this direction because Maxwell viscoelasticity causes the maximum displacement to occur after the maximum stress. Thus since the ocean-imposed displacement field is oriented toward the parent planet, the stress pattern is shifted in the direction the shell has rotated from. Once again, because the core is synchronously rotating, the forcing is constant at every point in the core and so the core still responds as though it were a fluid.

7. Numerical method, and a complication due to the core

For our numerical calculations we consider a satellite composed of four homogeneous compressible layers: a rocky core, an overlying fluid ocean, and a viscoelastic outer icy shell with a stiff upper layer (representing a conductive stagnant lid) and a low-viscosity basal layer (representing a possible convective region).

We recognize that the very cold surface layer of the ice shell will behave as a brittle-plastic material (e.g., Dombard and McKinnon, 2006) rather than viscously even on very long timescales. Whatever effect this near-surface layer has on the surface stresses is thus not accounted for in this model. However, if the fractured layer of the satellite extends to depths at which viscoelastic deformation occurs on the timescale of NSR, then our model should provide an accurate representation of the stresses at that depth.

In this example we choose the ice shell to be thin compared to the satellite’s radius, but this is not a requirement of the method; it can be applied equally well to a satellite with a thick outer shell, or even to a satellite with no liquid ocean at all. We determine the diurnal and NSR tidal solutions by solving the equations of motion for a stratified, compressible, and self-gravitating body (Eqs. (6)–(8)). Our numerical method is based on standard
Fig. 1. The complex Love numbers $\tilde{h}_2$ (a) and $\tilde{\ell}_2$ (b). The dashed curves are calculations in which the silicate core has a nearly fluid response, appropriate to NSR (see Section 7). The solid curves indicate a core having a nearly rigid response, appropriate to the diurnal forcing. Imaginary parts are thin curves, tied to the right axes; real parts are thick curves, tied to the left axes. The upper shell’s complex Lamé parameters $\tilde{\lambda}$ (c) and $\tilde{\mu}$ (d). Thick dashed curves correspond to real parts and thin solid curves are the imaginary parts. See Section 8.1 for discussion. All plots use the parameters listed in Table 1, and a range of forcing frequencies $\omega$, represented here by the parameter $\Lambda$ which represents the value of $\Delta$ in the upper layer.
Fig. 1. (continued)
algorithms used by geophysicists to compute tides on the Earth, and involves a modified version of the code used by Dahlen (1976) to compute terrestrial tides (see also Wahr et al., 2006; Rappaport et al., 2008). We have modified this code to include complex rheological parameters and complex solution scalars, so that we can accommodate a Maxwell solid rheology. The diurnal and NSR solutions differ solely through the frequency-dependence of the Lamé parameters, as shown in Eqs. (21)–(22). Once we have determined the complex Love numbers, $h$ and $\ell$, they are used as described in Section 5 to find the surface stresses.

In the formulation of the model described above, we assume the coordinate system is attached to the icy shell and so rotates with it. The equations of motion (Eqs. (6)–(8)) depend on the assumption that all displacements are small as seen in this coordinate system. These are reasonable assumptions for the diurnal tides. However, in the case of NSR it is only the icy shell that rotates nonsynchronously; the rocky core likely remains synchronously locked to the satellite's orbital motion (Greenberg and Weidenschilling, 1984). From the perspective of points in the icy shell, which is the region of most interest to this study, the external gravity field changes partly because the shell is rotating through the parent planet's gravity field, and partly because the shell is rotating through the gravity field caused by the underlying core. At points within the rocky core, on the other hand, the gravity field never changes, and so as described in Section 6, the core always responds to the NSR tidal forcing as though it were a fluid.

To obtain an adequate representation of the shape of the core and its effects on the icy shell, we set $\mu \approx 0$ in the rocky core when solving for the NSR tides. This eliminates shear stresses in the core, leading to much larger core tidal displacements. This in turn significantly increases the displacements and shear stresses in the icy shell, since then the gravity field from the core's tidal bulge can be a much larger fraction of the direct gravity field from the parent planet. On Europa for example, setting $\mu \approx 0$ in the core can increase the displacements and shear stresses in the icy shell by up to 70%. This difference in the assumed behavior of the core under the diurnal and NSR forcings is why there are two sets of each of the Love numbers displayed in Figs. 1a and 1b.

8. Results

Our formulation of the equations of motion and the resulting stresses describes the effects of viscoelasticity in terms of the parameter $\Delta$, which is proportional to the ratio between the forcing period and the Maxwell relaxation time of the ice. Because the shell we are considering has two viscoelastic layers with different viscosities, it will also have two values of $\Delta$ for a given forcing period. The resulting stresses at the outer surface (Eqs. (29)–(31)) depend on those values in two ways:

(i) they depend on the $\Delta$ values of each ice layer through the Love numbers $h$ and $\ell$; and

(ii) they depend on $\Delta$ of the upper icy layer only, through $\beta_1$, $\gamma_1$, $\beta_2$, $\gamma_2$, and $\Gamma$.

Item (i) represents the effects of viscoelasticity on the surface displacements, whereas (ii) describes how those displacements translate into stresses within a viscoelastic medium (with the properties of the upper ice layer). The dependence (i) is weak so long as there is an underlying ocean and the ice shell is thin, as we are assuming in this application. $\Delta$ does have a large relative effect on the imaginary parts of the Love numbers, but the imaginary parts are only a small fraction of the real parts. The tidal response in this case is mostly determined by the ocean, and so the Love numbers are only weakly dependent on the properties of the ice shell (see Moore and Schubert, 2000; Wu et al., 2001; Wahr et al., 2006; Rappaport et al., 2008). Thus, the primary influence of viscous effects comes through (ii).

8.1. Love numbers

The effects of viscoelasticity on the real and imaginary parts of the Love numbers are shown in Figs. 1a and 1b, for values of $\Delta$ (at the outer surface) spanning nine orders of magnitude. Values of $\Delta \ll 1$ indicate a forcing period short enough that the shear stresses do not have time to relax during a forcing cycle, and so the material behaves nearly elastically. Values of $\Delta \gg 1$ imply a long enough forcing period that the stresses have time to almost completely relax, allowing the material to behave almost as an inviscid fluid.

The results shown in these figures are computed using the upper and lower viscosity values $\eta_{\text{upper}}$ and $\eta_{\text{lower}}$ shown in Table 1, and varying the forcing period. The elastic value of $\mu$, given in Table 1, is assumed to be the same in each layer. Thus, the results are computed assuming the values of $\Delta$ in the lower and upper layers are related by a factor of $\Delta_{\text{lower}}/\Delta_{\text{upper}} = \eta_{\text{upper}}/\eta_{\text{lower}} = 10^5$. We would have obtained the same results if we had fixed the forcing period and varied the viscosities of the layers in tandem (i.e. maintaining $\eta_{\text{upper}}/\eta_{\text{lower}} = 10^5$).

Results are shown both for the diurnal tides, where the period is well known but the viscosity is not, and for the NSR tides, where neither the period nor the viscosity is known. The difference between the diurnal and NSR results at any given value of $\Delta$ is because in the NSR case we assume the silicate core behaves as a fluid, and in the diurnal case we assume it behaves elastically. As described in Section 7, this causes the NSR Love numbers to be about 70% larger than the diurnal Love numbers for a fixed value of $\Delta$.

Figs. 1a and 1b show that the real parts of the Love numbers (thick lines, tied to left axes) are one to two orders of magnitude larger than the imaginary parts (thin lines, right axes) no matter what value is assumed for $\Delta$. This is because the icy lithosphere is thin and so has only a small impact on the surface displacements regardless of whether it is viscoelastic or not. For the same reason, the real parts of the Love numbers vary by only $\sim 10\%$ over this entire range of $\Delta$. This is important because the viscosity profile beneath the very outermost ice layer can perturb the surface stresses only through its effects on the Love numbers, and the figures show that those effects are likely to be no larger than $\sim 10\%$. Thus there is little additional accuracy to be gained by including a more complicated internal viscosity structure in the shell.

Figs. 1a and 1b also show dips in the values of the imaginary parts of the Love numbers when $\Delta \approx 1$ and $\approx 10^{-5}$, with associated step function increases in the real parts. The $\Delta \approx 1$ features reflect the transition from elastic-like to fluid-like behavior in the outer shell, as $\Delta$ transitions between $<1$ and $>1$. These features result from similar behavior in the imaginary and real parts of $\mu$ and $\lambda$, of the outer shell evident in Figs. 1c and 1d. The features at $\Delta \approx 10^{-5}$ are due to a similar elastic-to-fluid transition in the lower ice layer: a value of $\Delta = 10^{-5}$ in the upper layer implies $\Delta = 1$ in the lower layer because we have chosen to fix the ratio $\eta_{\text{upper}}/\eta_{\text{lower}} = 10^5$.

There are also dips in the imaginary parts of the Love numbers at $\Delta \approx 300$, and corresponding step functions in the real parts. These features, which are far more prominent for the Love number $\ell$ than for $h$, represent the effects of an additional relaxation mode of the system, a mode with a relaxation time that is considerably longer than the Maxwell times of the ice layers. This situation is analogous to the post-glacial-rebound process on the Earth. Viscoelastic models of the Earth exhibit a large suite of relaxation modes. Some directly correspond to Maxwell times. Others, referred to as buoyancy modes, have longer periods and involve
radial displacements of density discontinuities. The $\Delta \approx 300$ mode evident in Figs. 1a and 1b corresponds to the viscoelastic mode usually referred to as CO in the post-glacial-rebound literature, associated with the relaxation of the Earth’s core-mantle boundary (e.g., Peltier, 1985). When the ocean/shell boundary is displaced from an equipotential surface, there is a gravitational (buoyancy) force that acts to restore it. This force is opposed by viscous resistance within the shell. Although this contribution to the viscoelastic Love numbers is interesting from a dynamical viewpoint, its impact on the surface stresses is minimal because the Love numbers play only a secondary role in determining the viscoelastic contributions to the surface stresses. However, it does cause the maximum phase shift to slightly exceed 45° when $\Delta > 1$, as can be seen in Fig. 2a for Europa. Reducing the density contrast between the ice and the ocean reduces the buoyancy force, and increases the timescale on which these forces have an effect, pushing both the large dip in the imaginary part of $\tilde{\epsilon}$, and the increase in phase shift to values greater than $45^\circ$, out to larger $\Delta$ values.

### 8.2. The direct effects of the outer layer’s viscosity

Viscous effects influence the surface stresses most through the Lamé parameters $\tilde{\mu}$ and $\tilde{\lambda}$ of the upper shell, and their impact on the parameters $\tilde{\rho}_1$, $\tilde{\gamma}_1$, $\tilde{\rho}_2$, $\tilde{\gamma}_2$, and $\tilde{\Gamma}$ (Eqs. (32)–(36)). The surface stress components (Eqs. (29)–(31)) are proportional to those last five parameters, and each of those parameters is proportional to $\tilde{\mu}$ of the outer surface. All except $\tilde{\Gamma}$ also depend on $\tilde{\lambda}$, but that dependence is not very strong. Figs. 1c and 1d show the Lamé parameters as functions of $\Delta$. Over the range of $\Delta$ considered here, $\tilde{\mu}$ varies enormously (from zero to $3.5 \times 10^{15}$ Pa) but $\tilde{\lambda}$ varies by only ~30% (from ~7 to ~9 x 10^5 Pa). This variability, combined with the direct influence of $\tilde{\mu}$ on all of the parameters listed above, implies that the $\Delta$ dependence of $\tilde{\mu}$ will strongly and directly impact the surface stresses.

Note from Eq. (24) that $\text{Im}(\tilde{\mu})/\text{Re}(\tilde{\mu}) = \Delta$. Thus when $\Delta \ll 1$ the imaginary part is small relative to the real part, and when $\Delta \gg 1$ it is large (though both the real and imaginary parts vanish as $\Delta \to \infty$). When the real and imaginary parts are of comparable magnitude (i.e. $0.1 < \Delta < 10$), $\tilde{\mu}$ departs significantly from either the fluid or elastic limits. All of these characteristics of $\tilde{\mu}$ get passed through directly to the stress components. The value of $\Delta$ at the outer surface thus becomes critical in determining the surface stresses.

For the diurnal tides, assuming a plausible small value for the viscosity of cold surface ice of $10^{22}$ Pas (Table 1), $\Delta$ in the upper layer is roughly $2 \times 10^{-8}$, which means the effects of viscoelasticity can be safely ignored. The outer surface viscosity would have to be as small as $2 \times 10^{15}$ Pas for $\Delta$ to be as large as 0.1, which is roughly when viscous effects start to become important. A viscosity of $2 \times 10^{15}$ Pas may be reasonable for the warm lower ice layer, but as described above, the lower layer $\Delta$ has only a minimal impact on the surface stresses, no matter its value. The implication is that viscoelastic effects are not likely to have a significant effect on the diurnal tides.

For the NSR tides (again assuming the outer layer viscosity is $10^{22}$ Pas), if the NSR period is between $1.2 \times 10^3$ and $1.2 \times 10^5$ yr, then $\Delta$ is in the range $0.1 < \Delta < 10$ where viscoelastic effects are important and are very sensitive to $\Delta$. If the NSR period is significantly longer than $1.2 \times 10^7$ yr, then $\Delta \gg 1$, and the surface stresses are small: they decay away almost as quickly as the forcing can create them.

However, even relatively small NSR stresses can overwhelm the diurnal stresses. The diurnal tides arise because of the orbital eccentricity $e$, and cause displacements that are $e$ times larger than the NSR displacements. Thus, if the ice layer was elastic, the diurnal stresses would similarly be a factor of $e$ times smaller than the NSR stresses. Viscoelasticity reduces the NSR stress magnitudes as $\Delta$ increases (i.e. as the NSR period gets longer). For $\Delta > 1$, the parameters $\tilde{\mu}_1$, $\tilde{\gamma}_1$, $\tilde{\rho}_2$, $\tilde{\gamma}_2$, and $\tilde{\Gamma}$ in Eqs. (32)–(36) become approximately inversely proportional to $\Delta$. The elastic case is given by those same expressions, but with $\Delta = 0$. Thus, in order for the amplitudes of the NSR stresses to be reduced to where they are comparable or smaller than the diurnal amplitudes, $\Delta$ has to be on the order of $1/e$ or larger (~100 for Europa). For an outer layer viscosity of $10^{22}$ Pas, an NSR period of ~$1.2 \times 10^8$ yr or longer is required for diurnal stresses to dominate NSR stresses.

### 8.3. Stress patterns and phase shifts

The value of $\Delta$ is critical for determining not only the magnitude of the surface stresses, but also how those stresses are distributed over the surface of the satellite. The smaller the value of $\Delta$, the closer the shell’s response is to being elastic, and so the closer the stresses are to being oriented symmetrically about the planet–satellite vector – a state we will refer to as having zero phase shift.

For the diurnal tides, $\Delta$ is so small that the resulting stresses are virtually elastic (see above). If the default viscosity values given in Table 1 are altered such that the entire shell has the high viscosity of the surface ($10^{22}$ Pas), the amplitude of the diurnal component of the stresses changes by less than 0.1%. If the entire low viscosity portion of the shell is replaced with ocean, leaving only an 8 km thick high-viscosity upper shell, the difference in the amplitude of the diurnal component of the stresses is ~3%. Maps of the diurnal stresses at different points in the orbit (i.e. at different values of $n$ in Eqs. (29)–(31)) are shown in Fig. 3. In these plots, the sub-jovian point at periarge is at latitude (y-axis) 0°, longitude (x-axis) 0°.

For the NSR tides the value of the period is unknown. Both the amplitude and the phase shift of the stresses depend on that period, mostly through the direct dependence on $\Delta$ of the outer surface. The parameters $\tilde{\mu}_1$, $\tilde{\gamma}_1$, $\tilde{\rho}_2$, $\tilde{\gamma}_2$, and $\tilde{\Gamma}$ (Eqs. (32)–(36)) are proportional to $\tilde{\mu}$, and $\tilde{\mu}$ depends on $\Delta$ through (24). Thus, the right-hand sides of Eqs. (32)–(36) show that when $\Delta < 1$ the parameters $\tilde{\rho}_1$, $\tilde{\gamma}_1$, $\tilde{\rho}_2$, $\tilde{\gamma}_2$, and $\tilde{\Gamma}$ are nearly real and are well approximated by their elastic values (since $\tilde{h}$ and $\tilde{\epsilon}$ are nearly equal to their elastic values).

When $\Delta > 1$, those five parameters have small amplitudes, but their imaginary parts are a factor of $\Delta$ larger than their real parts (for example, $1/(1 - i\Delta) = (1 - i\Delta)/(1 + \Delta^2)$, has real and imaginary parts of approximately $\Delta^{-2}$ and $\Delta^{-1}$ respectively, when $\Delta > 1$). This means that while $t_{nsr}$ is proportional to $\cos(2\phi + 2bt)$ in the elastic limit, it is nearly proportional to $\sin(2\phi + 2bt)$ in the $\Delta > 1$ limit (see Eq. (38)). The factors of 2 in $(2\phi + 2bt)$ mean that the spatial pattern of $t_{nsr}$ in the $\Delta > 1$ limit is displaced from the elastic pattern by $45^\circ$. The same is true of the other stress components $t_{\phi\phi}$ and $t_{\phi\theta}$.

In general the patterns and amplitudes of the tidally induced surface stresses can be separated into 4 possible regimes. If the orbital eccentricity $e$ is small and the shell is thin, and assuming for the case of Europa that the viscosity of the upper layer of the ice is $\eta_{upper} = 10^{22}$ Pas, we find:

(a) $\Delta < 0.1$. The NSR stress is nearly elastic, and so has a phase shift of ~0°. The NSR stress amplitude is a factor of ~$e^{-1}$ times larger than the diurnal stress amplitude. For our nominal Europa, this corresponds to an NSR period ($P_{nsr}$) of less than $1.2 \times 10^5$ yr, and tensile NSR stresses of up to ~3.2 MPa.

(b) $0.1 < \Delta < 10$. The NSR stress amplitude is still much larger than that of the diurnal stress, but it varies rapidly through this range of $\Delta$ values (for Europa the NSR stress amplitude
Fig. 2. (a) NSR results from our viscoelastic model. Shown are the amplitude and phase shift of the maximum tensile stress at the outer surface of Europa as a function of $\Delta$. The viscoelastic stress field experiences a phase shift $\geq 45^\circ$ for large values of $\Delta$ because of the effect the ice shell’s buoyancy mode has on the imaginary part of the Love number $\tilde{\ell}$ for values of $\Delta \approx 300$ (see Fig. 1b, and discussion in Section 8.1). If the density contrast between the ice and the ocean is reduced, this upturn moves out to larger values of $\Delta$, as does that feature in $\tilde{\ell}$. (b) Similar to (a), with accumulated degrees of NSR for the flattening model (which we compare our model to in Section 9) along the x-axis. (c) Because the variables that the two models use to determine the phase shift and stress amplitudes are different ($\Delta$ for the viscoelastic model, accumulated NSR for flattening), it is useful to compare the amplitudes as functions of the phase shift. The two curves are related by a nearly constant multiplier of $\sim 1.5$, with the flattening model predicting larger amplitudes for an identical amount of phase shift. As the phase shift approaches $45^\circ$ (implying small stress amplitudes), this multiplicative relationship breaks down, with the amplitude of viscoelastic stresses eventually exceeding that of the flattening stresses.
decreases from $\sim 3.2$ MPa to $\sim 500$ kPa as $\Delta$ increases. The phase shift varies from $\sim 0^\circ$ at $\Delta = 0.1$, to $\sim 45^\circ$ at $\Delta = 10$. For Europa this corresponds to $1.2 \times 10^5 < P_{\text{nsr}} < 1.2 \times 10^7$ yr. (c) $10 < \Delta < 1/e^1$ ($\approx 100$). The amplitude of the NSR stress has relaxed significantly, but is still as large or larger than that of the diurnal stress. The phase shift of the NSR stress pattern is $\sim 45^\circ$. For Europa this corresponds to $1.2 \times 10^7 < P_{\text{nsr}} < 1.2 \times 10^9$ yr, and results in the maximum tensile stress being reduced from $\sim 500$ kPa to $\sim 50$ kPa as $\Delta$ increases through this range of values. (d) $\Delta > 1/e^1$ ($\approx 100$). The amplitude of the NSR stress is smaller than that of the diurnal stress, becoming much smaller as $\Delta \rightarrow \infty$. The phase shift of the NSR stresses remains constant at $\sim 45^\circ$. However, for $\Delta \gg 1/e^1$ this becomes irrelevant after combining the NSR and diurnal stresses, because the NSR stress is overwhelmed by the diurnal stress. For Europa this corresponds to $P_{\text{nsr}} > 1.2 \times 10^8$ yr.

These points are illustrated in Fig. 4, which shows NSR stresses for different values of $\Delta$. The maps show the stress results (Eqs. (38)–(40)) evaluated at time $t = 0$. At any time $t$, NSR would cause the sub-jovian point at be at longitude $\phi = -br$ relative to the European surface. Thus, these maps can also be interpreted as showing the surface stresses at any time, relative to a sub-jovian point at latitude (y-axis) $0^\circ$ and longitude (x-axis) $0^\circ$.

In the case of Europa, it is not yet possible to determine which of the above regimes is appropriate. All four are possible given the current uncertainty in the NSR period and the range of plausible near-surface ice viscosities.

8.4. Geological implications of viscoelasticity

Attempts have been made to correlate lineaments on Europa with the NSR stress field by translating them longitudinally (Hoppe et al., 2001; Kattenhorn, 2002; Hurford et al., 2007; Figueredo and Greeley, 2000). The amount of translation required to get a good fit between a lineament and the stress field has been used as a proxy for the time elapsed since formation. The underlying assumption is that they formed at one longitude and, as the shell underwent NSR, they came to be located at another. This is only straightforward if the stress field is fixed with respect to the planet–satellite vector and has a known phase shift (previously assumed to be $\sim 45^\circ$). Because $\Delta$ can affect the phase shift of the NSR stresses, the apparent longitude of formation of a lineament will also depend on $\Delta$. If $\Delta$ is constant through time, this could introduce a constant translation of up to $45^\circ$ to the lineaments whose shapes are determined by the NSR stress field. But their apparent longitudes of formation could still potentially be used as a proxy for their relative times of formation. However, if we allow that $\Delta$ may have changed through time (Nimmo et al., 2006), it becomes difficult to infer even a relative time of formation, since different lineaments could have formed under different NSR stress regimes having different phase shifts.

Moreover, a variable $\Delta$ could allow the tidal stresses to transition between being dominated by diurnal and NSR tides. This could explain why on Europa we see both cycloidal lineaments (so far best explained by the diurnal tides) and the long arcuate global lineaments (so far best explained by the NSR tides). If a change in $\Delta$ reflects a change in the rate of NSR, it could be that the global lineaments were formed during a period of rapid rotation in which the shell responded elastically to the NSR tide; and the cycloidal lineaments during a period of slow shell rotation, in which the NSR stresses were able to relax viscously, leaving the diurnal tide to dominate. Alternatively, a change in ice shell viscosity, e.g. through an episode of increased tidal heating or intense convection, could produce a similar change in $\Delta$ over time.

The model described here assumes the rate of NSR is constant. It assumes the time-dependence of the NSR forcing can be repre-
Fig. 3. Diurnal stress results for Europa from the viscoelastic model, at different orbital locations, measured from perijove ($nt$ in Eqs. (29)–(31)), using parameters from Table 1. The sub-jovian point at perijove is at latitude ($\gamma$-axis) 0°, longitude ($\phi$, $x$-axis) 0°. East is taken as positive. Results for $180^\circ < \phi < 360^\circ$ are the same as those between 0° and 180°: $\tau(\phi) = \tau(\phi + 180^\circ)$. Stresses in the second half of the orbit ($180^\circ < nt < 360^\circ$) are east–west reflections of those in the first half. Tic marks show the magnitude and orientation of the principal components of the stresses on the surface of the satellite. Compression (blue) is negative and tension (red) is positive. Background color shows the magnitude of the most tensile of the two principal components, as indicated by the color scales shown to the right of each panel.

sented as a sinusoidal function with a single frequency. For an NSR rate that varies with time, the stresses could be computed by expanding the NSR forcing as a sum of sinusoidal functions, using the model described in this paper to find the stresses caused by each of those sinusoidal functions, and then summing those stresses together. A change in NSR would cause changes in the stress that depend not only on the initial and final NSR rates, but also on how quickly the NSR evolved. If that change in rate happened quickly the induced stresses could be large during the transition even if the initial and final NSR rates were slow.

9. Comparison to previous methods

Results from the viscoelastic model described here can be used to assess the “flattening” model used in previous work to predict both diurnal and NSR stresses (Helfenstein and Parmentier, 1985; Leith and McKinnon, 1996; Hoppa, 1998; Greenberg et al., 1998; Hurford et al., 2007).

To estimate diurnal stresses, the flattening model computes the elastic response to the diurnal tidal potential terms, Eqs. (4) and (5). To estimate NSR stresses, the model takes the difference between two stress fields. Each field is the elastic stress pattern caused by the NSR potential Eq. (3), with one field rotated about the $\hat{z}$ axis relative to the other. In effect, one stress field is computed for $bt = 0$, and the other for $bt$ = some specified angle.

In both the diurnal and NSR cases, the flattening model relates the elastic stress field to the elastic Love number $h$; either explicitly (Hurford et al., 2007), or implicitly through a flattening parameter (Leith and McKinnon, 1996). Although the Love number $\ell$ does not occur explicitly in the flattening formalism, a comparison with our elastic results, Eqs. (12)–(14), shows that the flattening model
Fig. 4. NSR stress results from the viscoelastic model for Europa at different values of $\Delta$. Plots are similar to those in Fig. 3. The sub-Jovian point is at latitude $= 0^\circ$, longitude $= 0^\circ$. This pattern is repeated for $\phi > 180^\circ$, $\tau(\phi) = \tau(\phi + 180^\circ)$. The tick mark length and color scales indicating stress magnitudes vary among the four panels. Results are computed using the input parameters shown in Table 1, with the exception of the orbital eccentricity $\epsilon$, which has been set to zero to exclude contributions from diurnal stresses. (a) $\Delta = 0.1$, corresponding to nearly elastic stresses. The stress field is close to symmetric about the planet–satellite vector (which passes through $0^\circ$ and $180^\circ$ longitude), though a small westward shift in the stresses is evident. (b) $\Delta = 1$. The magnitudes of the stresses have been reduced slightly by relaxation, but the westward phase shift has increased significantly. (c) $\Delta = 10$. The NSR stresses have largely relaxed away, though they are still significantly larger than the diurnal stresses pictured in Fig. 3. Here the phase shift is nearly complete, with the region of greatest tensile stress close to its maximum separation from the planet–satellite vector. (d) $\Delta = 100$ ($\sim \epsilon^{-1}$ for Europa). The overall NSR stress pattern is persistent, but the magnitudes are significantly smaller than those in Fig. 3. See Section 8.3 for further discussion.

implicitly assumes $\ell = h/4$, a result that is in good agreement with the elastic Love number results found here (see Figs. 1a and 1b in the $\Delta \to 0$ limit).

Figs. 5a and 5b compare our diurnal stress results with those of the flattening model applied to a similar satellite. Both sets of results were computed for a time corresponding to $225^\circ$ after perijove (i.e. $nt = 225^\circ$ in Eqs. (29)–(31)). The two methods predict similar overall patterns. Amplitude differences are on the order of only 7%, and presumably reflect differences between the interior models used.

The situation is more complicated for the NSR stresses. The flattening method’s representation of those stresses as the difference between two elastic stress fields is somewhat ad hoc, and its relationship to the viscoelastic properties of the shell is not immediately obvious. Fig. 2 can be used to empirically understand that relationship.

Fig. 2a shows the largest tensile stress magnitude and the phase shift for the viscoelastic NSR model, as a function of $\Delta$ in the upper ice layer. The phase shift is defined here as the number of degrees of longitude separating the greatest tensile stress and the planet–satellite vector. For an elastic satellite the phase shift $= 0^\circ$. Large values of $\Delta$, corresponding to slow NSR rates and/or short viscous relaxation times, lead to phase shifts of $45^\circ$ and small stress amplitudes.
Fig. 5. Stresses computed using the “flattening” method (left) compared to those from our viscoelastic model (right) for a Europa-like satellite with similar physical properties. Table 4 lists the input parameters adopted for the flattening calculations (T. Hurford, personal communication, 2006). These parameters, excluding the real valued Love numbers, have also been adopted for the viscoelastic calculations. Because the viscoelastic model requires more information about the internal structure of the satellite in order to calculate the frequency-dependent Love numbers, those parameters in Table 1 not listed in Table 4 have been adopted in (b) and (d). (a) Flattening and (b) viscoelastic results for diurnal stresses at 225° after perijove (i.e. nt = 225° in Eqs. (29)–(31)). (c) NSR stresses from the flattening method with 1° of accumulated NSR, compared to those calculated by the viscoelastic model (d) with Δ = 56, at which point the two models each have a phase shift of 44.5°. See Section 9 for discussion.

Table 4
Flattening model parameters (Figs. 2 and 5).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic Love number</td>
<td>h</td>
<td>1.2753</td>
</tr>
<tr>
<td>Elastic Love number</td>
<td>ℓ</td>
<td>0.31882 (≈ \frac{1}{3})</td>
</tr>
<tr>
<td>Mass of Europa</td>
<td>ME</td>
<td>4.80 \times 10^{22} kg</td>
</tr>
<tr>
<td>Mass of Jupiter</td>
<td>M_p</td>
<td>1.8986 \times 10^{27} kg</td>
</tr>
<tr>
<td>Radius of Europa</td>
<td>Rs</td>
<td>1.561 \times 10^{10} m</td>
</tr>
<tr>
<td>Europa’s orbital semi-major axis</td>
<td>a</td>
<td>6.709 \times 10^{11} m</td>
</tr>
<tr>
<td>Eccentricity of orbit</td>
<td>e</td>
<td>0.01</td>
</tr>
<tr>
<td>Bulk modulus of ice</td>
<td>\lambda_{ice} + \frac{2}{3} \mu_{ice}</td>
<td>9.1764 \times 10^8 Pa</td>
</tr>
<tr>
<td>Shear modulus of ice</td>
<td>\mu_{ice}</td>
<td>3.5187 \times 10^9 Pa</td>
</tr>
</tbody>
</table>

Fig. 2b shows the same quantities for the flattening model, as a function of the rotation angle between the two elastic stress patterns (the “accumulated degrees of NSR,” which in the flattening model is the parameter that describes how much stress is allowed to build up in the shell, similar to our Δ). Small values of the rotation angle result in large phase shifts and small amplitudes. This can be understood by considering, for example, \xi_{\phi \phi}. The elastic result (Eq. (12)) has a longitudinal dependence of \cos(2\phi + 2b\theta). The difference between this cosine and the same cosine when \phi has been rotated by the angle \delta, is \cos(2\phi + 2b\theta) - \cos(2\phi + 2(\phi + \delta + b\theta)) \approx (when \delta is small) \delta \sin(2\phi + 2b\theta). Thus, the difference decreases in amplitude as \delta \to 0, and lags \cos(2\phi + 2b\theta) by 45° in \phi. The difference between these cosines has an increasing amplitude and a decreasing phase shift as \delta increases, consistent with the results shown in Fig. 2b.
Fig. 2c compares our viscoelastic results with the flattening results, by plotting the maximum stress as a function of the phase shift. The general shapes of the viscoelastic and flattening curves are similar. Though for a given phase shift the flattening model tends to overestimate the maximum stress amplitude by a factor of about 1.5. Alternatively, for a given maximum stress the flattening model predicts a larger phase shift. The results thus show that the viscoelastic and flattening models predict different relationships between the magnitude of the NSR stresses and their location on the surface of the satellite. Either the phase shift or the magnitude of the NSR stresses can match, but not both simultaneously. Note that had we not allowed shear stresses to relax in the synchronously rotating core by using a low value of $\mu_{\text{core}}$ in the NSR Love number calculations, as discussed in Section 7 above, the real parts of the NSR Love numbers would have been similar to the diurnal values (see Figs. 1a and 1b, and Tables 2 and 3), resulting in the viscoelastic stresses being $\sim 20\%$ smaller, and increasing the difference from the flattening model even more.

To compare spatial stress patterns for the two different models of NSR stresses, we need to match up values of $\Delta$ in the viscoelastic model with corresponding values of accumulated NSR in the flattening model. The results shown in Figs. 2a and 2b indicate that small values of accumulated NSR correspond to large values of $\Delta$ (i.e. to long NSR periods and/or small viscosities). For every value of accumulated NSR in the flattening model we find the corresponding value of $\Delta$ in the viscoelastic model such that the two models give identical phase shifts for the maximum tensile stress. Because of the linear relationship between phase shift and the amount of accumulated NSR evident in Fig. 2b, we expect $1^\circ$ of NSR to result in a phase lag of $44.5^\circ$ ($= 45^\circ - (0.5 \times \text{NSR})$). For the parameters used in these calculations, this corresponds to $\Delta = 56$. The comparison can be seen in Figs. 5c and 5d. The spatial patterns are in good agreement. The amplitudes of the flattening stresses are larger than the amplitudes of the viscoelastic stresses, but only by $\sim 20\%$ rather than the $\sim 50\%$ that might be expected from Fig. 2c. This is because Figs. 5c and 5d consider a phase shift close to $45^\circ$, which is where the flattening model has usually been applied in the past. Fig. 2c shows that as the phase shift gets close to $45^\circ$ the amplitude of the viscoelastic stresses approaches and eventually even exceeds that of the flattening stresses, though both are small at large values of $\Delta$. This is partly due to the buoyancy mode described at the end of Section 8.1, which begins to influence the surface stresses when $\Delta$ exceeds $\sim 10$ (see Figs. 1a, 1b, and 2a). The flattening stresses reach zero amplitude when the phase shift is $45^\circ$, but the viscoelastic stresses maintain an amplitude of at least a few kPa until the phase shift is close to $50^\circ$, and vanish only as $\Delta \rightarrow \infty$.

10. Summary and future work

We have developed and implemented a method of calculating the tidally induced surface stresses of a radially stratified satellite with a Maxwell viscoelastic shell of arbitrary thickness overlying an inviscid ocean and a silicate core, derived directly from the time-varying gravitational potential experienced by the satellite. All regions of the satellite are compressible and self-gravitating. The formalism could easily be extended to also include viscoelasticity within the silicate core, though we have chosen not to do so here. The results could also readily be extended to find the stress field at any depth within the shell, by using output from the numerical Love number code at subsurface depths.

We have applied this model to radial and librational diurnal tides caused by the eccentricity of the satellite’s orbit, and to tides that would be caused by faster than synchronous rotation of a floating shell. In both these cases we assumed the satellite’s orbit has zero obliquity, so that the orbital motion is in the satellite’s equatorial plane, and that the NSR motion occurs in that same plane.

Viscoelastic effects are incorporated through the use of frequency-dependent, complex-valued Lamé parameters and Love numbers. The inclusion of viscous relaxation has significant implications for the NSR stress environment at the satellite’s surface, both reducing the magnitude of stresses due to long period forcings, and inducing a phase shift that translates the NSR stress field in the opposite direction of shell rotation. The importance of these effects depends on the ratio of the NSR period to the viscous relaxation time of the satellite’s outer surface, a ratio described here by the parameter $\Delta$. If $\Delta \leq 10$, NSR stresses are much larger than diurnal stresses, and are very similar to the elastic limit with a $\sim 0^\circ$ phase shift. If $\Delta \geq 100$, the NSR stresses will have a phase shift of $\sim 45^\circ$, but their amplitude will be smaller than the diurnal stresses. The effects of viscoelasticity on the diurnal stresses are insignificant for any plausible value of outer surface viscosity.

Because $\Delta$ affects the phase shift of the stress field, the apparent longitude of formation of a lineament will also depend on $\Delta$. If we accept the possibility that $\Delta$ changes through time, this makes it more difficult to use a lineament’s apparent longitude of formation as a proxy for its time of formation, even relative to other lineaments, since they may have formed under NSR stress regimes with different phase shifts.

If we think the linear features observed on the surface of an icy satellite are tidally induced or influenced fractures, it must follow that the surface stresses sometimes exceed the strength of the icy lithosphere. This implies that localized stress release due to brittle failure plays a role in defining the surface stress environment (Smith-Konter and Pappalardo, 2008). It would be beneficial to incorporate the formation of brittle fractures and the resulting changes in the stress field into the viscoelastic model (cf. King et al., 1994). However, that modeling is inherently numerical, requiring localized adjustment of the stresses as each crack forms and affects the formation of subsequent fractures in the region. Thus, the stresses of our model are those one would expect to find on the surface of a viscoelastic shell stronger than the greatest calculated stress. In this paper we have applied this model to the stresses experienced by a shell in steady state with a constant rotation rate, but there are many other possible scenarios for re-orientation of a decoupled shell that are not well represented by a steady-state solution. A time-variable NSR rate can easily be accommodated using the formalism described here, while calculating the time evolution of stresses due to episodic polar wander will require enhancements to the model.

11. Sharing the model with the community

Other researchers are encouraged to create their own implementation of the model described in this paper. For those who prefer to use, verify, or build upon our implementation of the model, we are providing access to the code under a public license at: http://code.google.com/p/satstress.

We are also providing a web-based interface to the model at: http://icymoons.com/satstress where users may input model parameters, and perform regularly gridded calculations like those used to generate the figures presented in this paper.

An important benefit of hosting the model on the web is that individual model runs can be archived automatically for later reference. For example, Table 5 contains the unique model run IDs of the SatStress calculations that went into making Figs. 3–5. With one of these IDs a user can view all the model inputs and outputs pertaining to the run. They can also use any run ID as the basis of a new model run. This centralized recordkeeping makes it easier to track and compare model inputs and outputs without having
Appendix A. Tidal potential

Let \((r, \theta, \phi)\) be the spherical coordinates \((r = \text{radius}, \theta = \text{co-latitude}, \phi = \text{longitude})\) of a point in the satellite. The general form of the tidal potential at \((r, \theta, \phi)\) caused by an external point mass (the parent planet), is given in Eq. (1) of Kaula (1964) (changing some of the variable names):

\[
V_T(r, \theta, \phi) = \frac{Gm^*}{a^*} \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \frac{r}{a^*} \left( \frac{l-m}{l+m} \right) \sin^2 \frac{\theta}{2} \left[ \cos l \sin \theta \sum_{\ell=0}^{l} \left( \frac{\ell}{2} \right) P_{lm}(\cos \phi) G_{pq} \epsilon \right]
\]

where the six Keplerian elements that describe the parent planet’s apparent motion are \(\Omega^*\) (longitude of the ascending node), \(M^*\) (mean anomaly), \(i^*\) (inclination), \(a^*\) (semi-major axis), \(\omega^*\) (argument of pericenter), and \(e\) (eccentricity). Other variables are: \(\theta^*\) is the sidereal time of the reference meridian, \(m^*\) is the mass of the planet, \(G\) is Newton’s gravitational constant, the \(P_{lm}(\cos \phi)\) are associated Legendre functions, and the \(P_{lm}(t^*)\) and \(G_{pq}(\epsilon)\) are polynomials given in Tables 2 and 3 of Kaula (1964). For most satellites of interest \(r/a^* \ll 1\). For Europa, for example, \(r/a \leq 0.0023\). Thus, as usual, we keep only \(l = 2\) terms in \(A.1\). We set \(i^* = 0\) (since we are assuming the satellite’s obliquity vanishes), which causes the only non-zero \(F_{2mp}(t^*)\) to be \(F_{220} = 3\), and \(F_{201} = -1/2\). We assume the eccentricity is small \((e \sim 0.0094\) for Europa), and expand \(G_{pq}(\epsilon)\) in powers of \(e\) keeping terms only up to first order. We set \(M^* = nt\) (which is exact for Keplerian orbits), and assume \(\theta^* = (n + b) t\) \((\theta^*\) represents the angular displacement between the vector to the planet and the \(x\)-axis in our satellite-fixed coordinate system). By choosing \(M^* = nt\), we are defining \(t = 0\) as the time of periapsis. Because the inclination vanishes, there is no distinction between \(\Omega^* + \omega^*\), and only the sum \((\Omega^* + \omega^* + \epsilon\) appears in the final solution. We set this sum to zero, which, together with our assumption that \(\theta^* = 0\) at \(t = 0\), implies that not only is the satellite at periapsis at time \(t = 0\), but its \(x\)-axis is pointing toward the planet at that time. The longitude, \(\phi, \phi\) of a point in the satellite is defined as the angle eastward (i.e. counterclockwise as seen above the rotation axis) from this \(x\)-axis. Using expressions for the \(P_{lm}(\cos \phi)\) gives:

\[
V_T(r, \theta, \phi, t) = \frac{Gm^* R_s^2}{2a^*} \left[ T_s + T_0 + T_1 + T_2 \right],
\]

given that

\[
T_s = \frac{1}{6} \left( 1 - 3 \cos^2 \theta \right),
\]

\[
T_0 = \frac{1}{2} \sin^2 \theta \cos(2\phi + 2bt),
\]

\[
T_1 = \frac{1}{2} \left( 1 - 3 \cos^2 \theta \right) \cos(nt),
\]

\[
T_2 = \sin^2 \theta \left[ 3 \cos(2\phi) \cos(nt) + 4 \sin(2\phi) \sin(nt) \right].
\]

Appendix B. Surface stresses for elastic satellites

The stress tensor associated with the tidal displacement vector, is given by Eq. (8) in the main text:

\[
\tau = \lambda(\nabla \cdot \mathbf{s}) + \mu \left[ \nabla \mathbf{s} + (\nabla \mathbf{s})^T \right].
\]

We use this relation, together with the surface displacement components first defined in Eqs. (9)-(11):

\[
s_r(r = R_s, \theta, \phi, t) = \left( \frac{g}{\theta} \right) \nabla V_T |_{r=R_s},
\]

\[
s_\theta(r = R_s, \theta, \phi, t) = \left( \frac{\epsilon}{g \sin \theta} \right) \frac{\partial V_T}{\partial \phi} |_{r=R_s},
\]

\[
s_\phi(r = R_s, \theta, \phi, t) = \left( \frac{\epsilon}{g} \right) \frac{\partial V_T}{\partial r} |_{r=R_s},
\]
and the tidal potential Eq. (1), to find $r$ at the outer surface as follows.

As noted in Section 4, $\tau_{rr} = \tau_{\theta \theta} = \tau_{\phi \phi} = 0$ at the outer surface. Writing $[V^n V (V^3)^{-1}]$ and $\nabla \cdot \mathbf{s}$ in spherical coordinates, as given by Eqs. (139) and (140) of Dahlen and Tromp (2001), we find that:

$$\tau_{rr} = - \frac{1}{R_0} \left[ R_0 \beta_s \gamma_s + 2 \beta_s \beta_0 + s_s \cot \theta + \frac{\beta_s \beta_0}{\sin^2 \theta} \right] + 2 \beta_s \beta_0 = 0. \quad (B.5)$$

This implies that at the outer surface (where $\tau_{rr} = 0$):

$$\partial_s \gamma_s = - \frac{1}{R_0} \left[ R_0 \beta_s \gamma_s + 2 \beta_s \beta_0 + s_s \cot \theta + \frac{\beta_s \beta_0}{\sin^2 \theta} \right]. \quad (B.6)$$

so that (using Eq. (B.6) in Eq. (140) of Dahlen and Tromp, 2001):

$$\nabla \cdot \mathbf{s} = \frac{2 \beta_s}{R_0} \left[ 2 \beta_s \beta_0 + s_s \cot \theta + \frac{\beta_s \beta_0}{\sin^2 \theta} \right]. \quad (B.7)$$

The only non-zero stress components at the outer surface are $\tau_{\theta \theta}$, $\tau_{\phi \phi}$, and $\tau_{\phi \theta}$. Using the spherical components of $[V^n V (V^3)^{-1}]$ given by Eq. (139) in Dahlen and Tromp (2001), and using Eq. (B.7) for $\nabla \cdot \mathbf{s}$, gives:

$$\tau_{\theta \theta} = \frac{2 \beta_s}{R_0} \left[ (3 \lambda + 2 \mu) s_s + 2 (\lambda + \mu) \beta_s \beta_0 \right] \left[ \frac{\beta_s \beta_0}{\sin^2 \theta} + s_s \cot \theta \right], \quad (B.8)$$

$$\tau_{\phi \phi} = \frac{2 \beta_s}{R_0} \left[ (3 \lambda + 2 \mu) s_s + \lambda \beta_s \beta_0 \right] \left[ 2 (\lambda + \mu) \beta_s \beta_0 + s_s \cot \theta \right], \quad (B.9)$$

$$\tau_{\phi \theta} = \frac{2 \beta_s \epsilon}{R_0} \left[ s_s \cot \theta + \frac{\beta_s \beta_0}{\sin^2 \theta} - s_s \cot \theta \right]. \quad (B.10)$$

At the outer surface, the displacements $s_s$, $s_0$, and $s_\phi$ can be related to the tidal potential, $V_T|_{R_s}$, evaluated at the outer surface, using the Love numbers $h$ and $\ell$ as described in (B.2)–(B.4). Using those results in Eqs. (B.8)–(B.10), gives:

$$\tau_{\theta \theta} = \frac{2 \beta_s}{g R_s} \left[ h((3 \lambda + 2 \mu) V_T|_{R_s}) \right] + \ell \left( \frac{\lambda}{\sin^2 \theta} \right) a^2 V_T|_{R_s} + \frac{2 (\lambda + \mu) \beta_s \beta_0}{\sin^2 \theta} V_T|_{R_s}, \quad (B.11)$$

$$\tau_{\phi \phi} = \frac{2 \beta_s}{g R_s} \left[ h((3 \lambda + 2 \mu) V_T|_{R_s}) \right] + \ell \left( \frac{2 (\ell + \mu)}{\sin^2 \theta} \right) a^2 V_T|_{R_s} + \frac{2 (\lambda + \mu) \beta_s \beta_0}{\sin^2 \theta} V_T|_{R_s}, \quad (B.12)$$

$$\tau_{\phi \theta} = \frac{2 \beta_s \epsilon}{g R_s} \left[ s_s \cot \theta + \frac{\beta_s \beta_0}{\sin^2 \theta} \right]. \quad (B.13)$$

Finally, using Eq. (1) to find $V_T|_{R_s}$ gives the results shown in Eqs. (12)–(14) of the main text:

$$\tau_{\theta \theta} = \frac{2 \beta_s}{g R_s} \left[ - \frac{1}{3} \left( \beta_1 + 3 \gamma_1 \cos(2\theta) \right) \cos(2\phi + 2bt) + \left( \beta_1 - \gamma_1 \cos(2\theta) \right) \cos(2\phi + 2bt) + 3e (\beta_1 - \gamma_1 \cos(2\theta)) \cos(2nt) \cos(2\phi) - \epsilon \left( \beta_1 + 3 \gamma_1 \cos(2\theta) \right) \cos(nt) \right] + 4e (\beta_1 - \gamma_1 \cos(2\theta)) \sin(nt) \sin(2\phi). \quad (B.14)$$


